When all holes in a graph have the same length

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Based on a joint work with:

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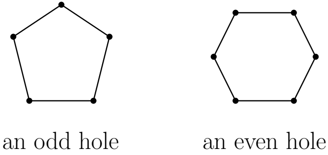
Holes in a graph

Hole = *chordless cycle* of length at least 4;

it is *even* or *odd* depending on the parity of its length.



A *subgraph* H of G is an *induced subgraph* if $uv \in E(H)$ iff $uv \in E(G)$.

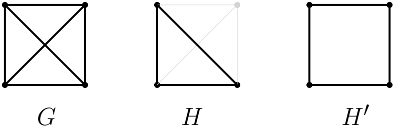
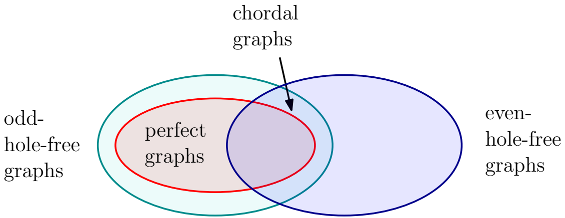


Figure: H is an induced subgraph of G, but H' is not

Well-known graph classes forbidding holes

1. **Chordal graphs** forbid all holes.
2. **Perfect graphs (Berge graphs)** forbid all odd holes (in the graph and in its complement).

   Perfect graphs form a subclass of odd-hole-free graphs.
3. **Even-hole-free† graphs** forbid all even holes.

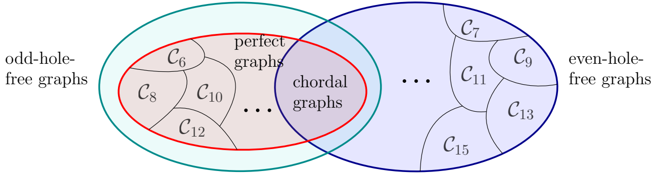


†being "H-free" means that it does not contain H as an *induced subgraph*

Problem statement

Motivating question:

What if all holes <u>*except those of a fixed length k*</u> *is forbidden?*



<u>Notation:</u>

For $k \geq 7$, $\mathcal{C}_k$ is the class of graphs, where every graph **contains** <u>**only**</u> **holes of length** k.

Objectives

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Thm: Structure of $\mathcal{C}_k^{\text{odd}}$

Every graph in $\mathcal{C}_k^{\text{odd}}$ either is:

- a ring of length k; or
- a blowup of an *odd* template;

or has either:

- a universal clique; or
- a clique cutset.

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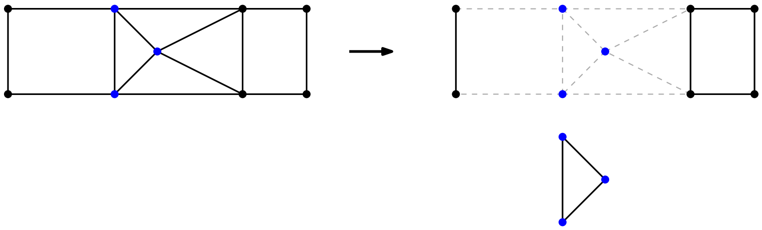
1. Clique cutsets

A *cutset* of a connected graph G is a set of vertices $S \subsetneq V(G)$ such that $G \setminus S$ is not connected. *Clique cutset* is a cutset that induces a complete graph.



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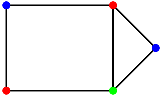

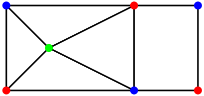

Why clique cutset?

- Attachment through a clique cutset does not create a new hole.
- Clique cutset is useful for divide-and-conquer approach for many algorithmic graph problems (*e.g., coloring problem*).

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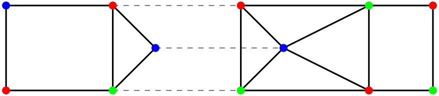

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2. Universal clique

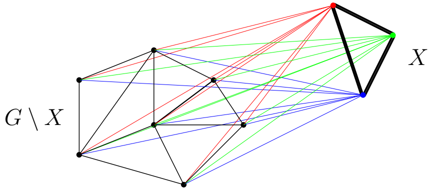


Figure: X is a universal clique in G

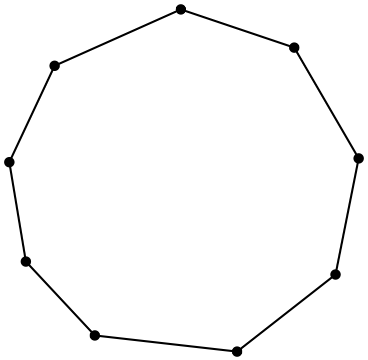
Implication of attaching a universal clique

Attaching a universal clique X to a graph G

does not create a new hole

(i.e., G and $G + X$ have the same set of holes).

3. Rings



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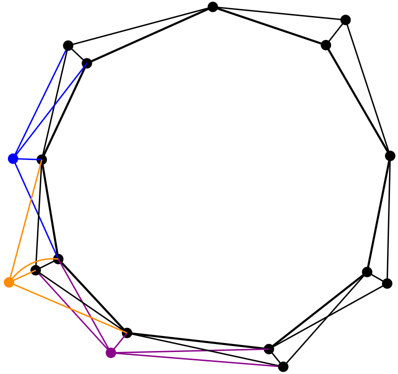

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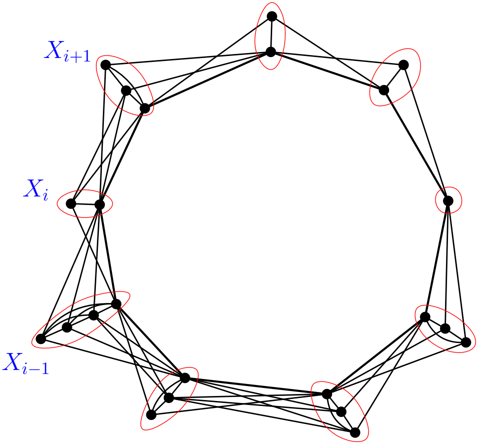

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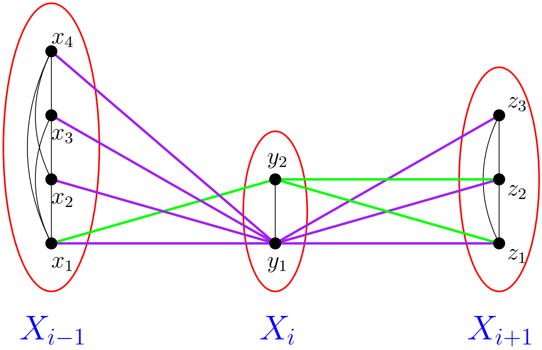

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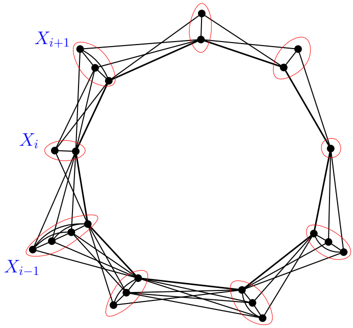
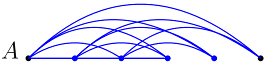


Remark:

- Rings of length k contain holes only of length k.
- Any hole is a ring.

4. *Odd* templates

Construction:



1. Create a *threshold* (i.e., (P_4, C_4, $2K_2$-free)) graph with vertex set $A = \{v_1, \ldots, v_t\}$.

Length = the number of vertices

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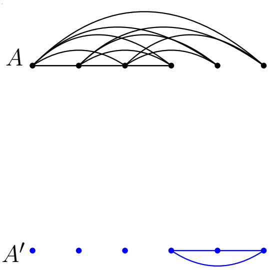

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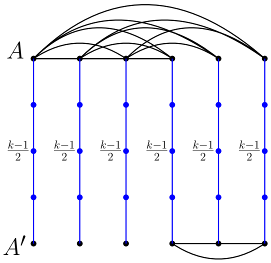

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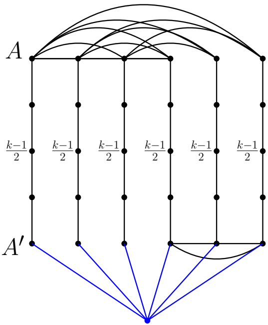

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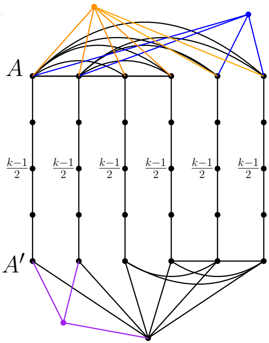

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5. Add some more vertices (possibly none) to some vertices of A (resp. A') by considering a certain type of hypergraph.

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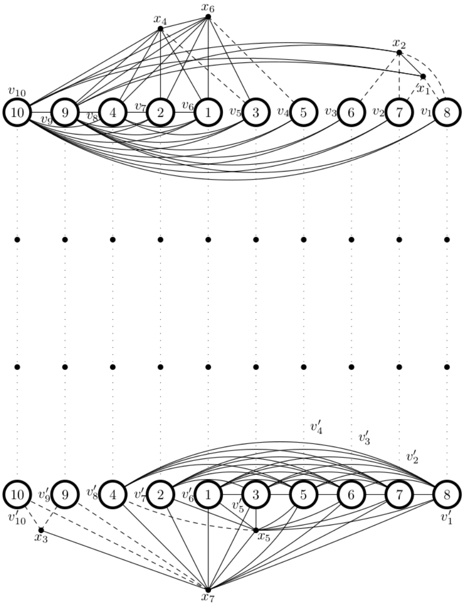

Even templates construction

1. Create a *threshold* graph with vertex set $A = K \cup S$ where $K = \{v_1, \ldots, v_t\}$ induces a clique and $S = \{v_{t+1}, \ldots, v_n\}$ induces a stable set. Similarly, create a threshold graph $A' = K' \cup S'$.

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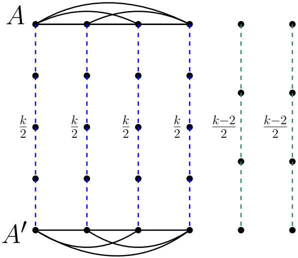

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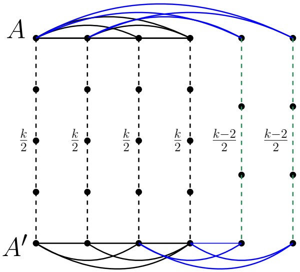

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3. Add some edges from S to A (resp. from S' to A'), where $v_i \in A$ and $v_j \in S$ are adjacent <u>if and only if</u> $v_i' v_j' \notin E(G)$.

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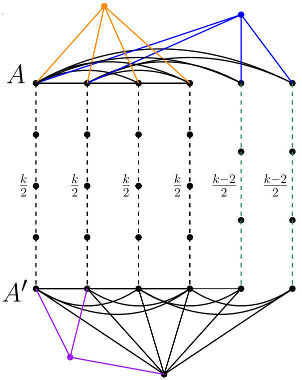

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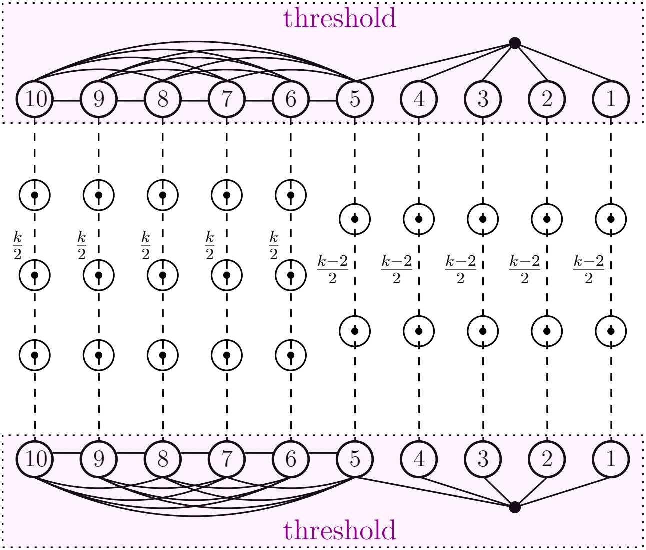


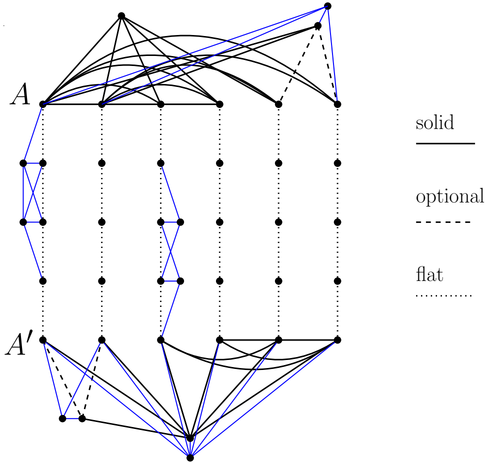
Blow-up of *odd* templates (even case is similar)

- Substitute each vertex $v \in V(G)$ with a clique K_v (of size ≥ 1).

Attachment between blowup of vertices:
(*three possible attachments*)

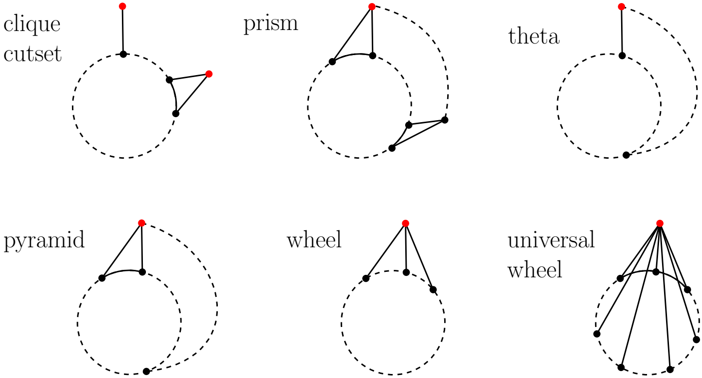
- For every $uv \notin E(G)$, K_u is anticomplete to K_v.
- For every $uv \in E(G)$, the adjacency between K_u and K_v are as in a ring.
- Additionally, for every edge uv, if the removal of uv from G yields a 'bad hole', then K_u is complete to K_v.

Blow-up of *odd* templates (even case is similar)



Idea of proof

Attachment on holes



Four configurations that appear when forbidding holes

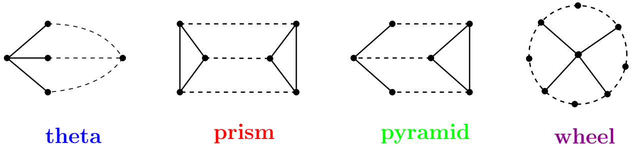
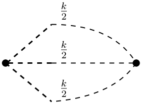


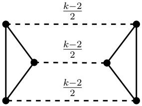
Figure: *4-configuration* (dashed lines represent paths of length at least 1)

Class	theta	prism	pyramid	wheel
Odd-hole-free graphs	✓	✓	×	✓
Even-hole-free graphs	×	×	✓	✓
Chordal graphs	×	×	×	×

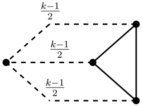
The four-configuration in $\mathcal{C}_k$



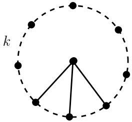
balanced theta



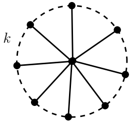
balanced prism



balanced pyramid



twin wheel

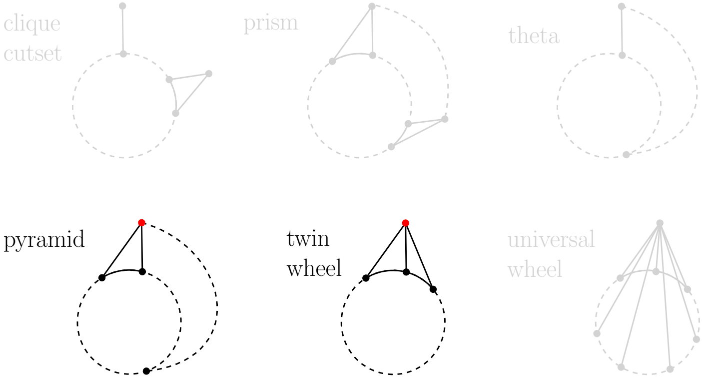


universal wheel

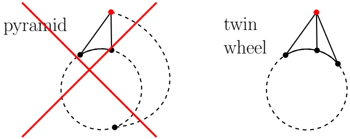
Proof 1: the class $\mathcal{C}_k^{\text{odd}}$

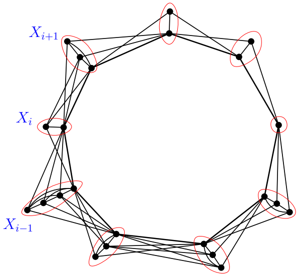
Attachment on holes:

Remark. Suppose that $G \in \mathcal{C}_k^{\text{odd}}$ does not have a *clique cutset* nor a *universal clique*.



Case 1: $\mathcal{C}_k^{\text{odd}}$, $k \geq 7$ when it is pyramid-free

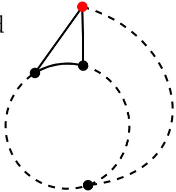




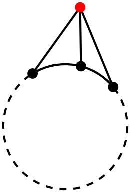
When there is *no pyramid*,
only odd rings possibly exist.

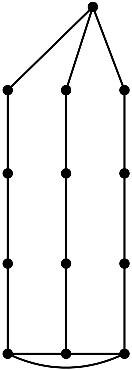
Case 2: $\mathcal{C}_k^{\text{odd}}$, $k \geq 7$ when there is a pyramid

pyramid



twin wheel





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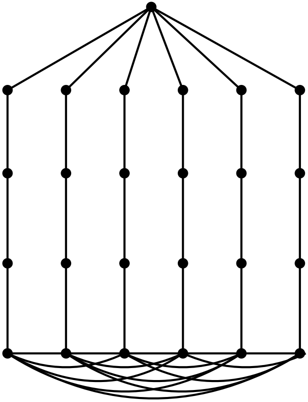

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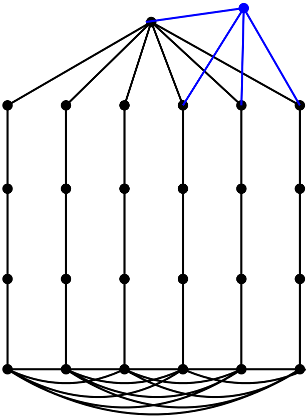

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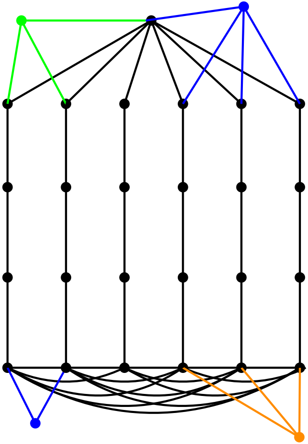

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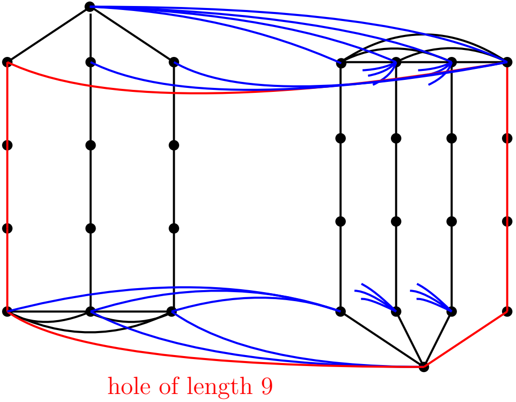

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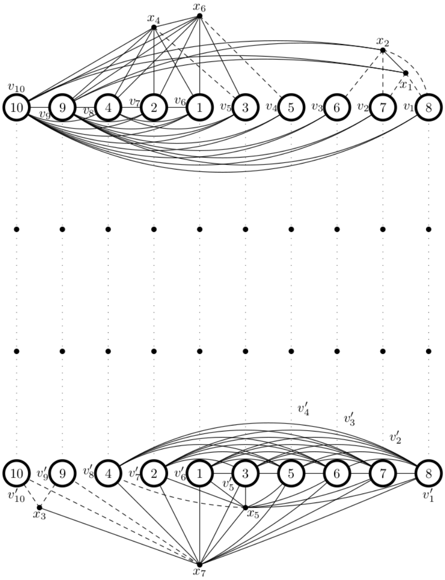

Why the top/bottom part is (P_4, C_4)-free?

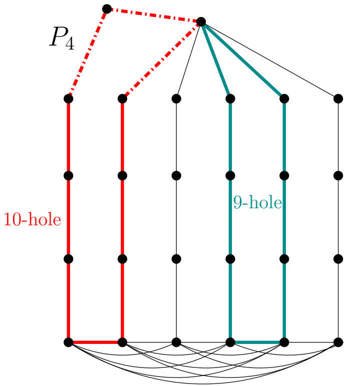

Figure: If the top/bottom part is **not** P_4**-free**, then there would exist a **"bad" hole**

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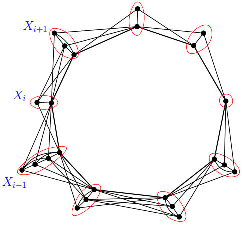

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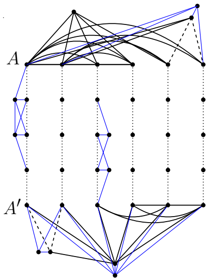
or has either:

- a universal clique; or
- a clique cutset.



Thm: Structure of $\mathcal{C}_k^{\text{odd}}$ ($k \geq 7$)

Every graph in $\mathcal{C}_k^{\text{odd}}$ either is:

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- a blowup of an odd template;

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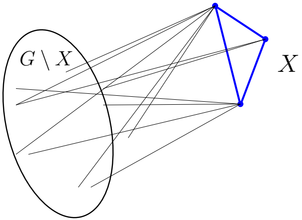

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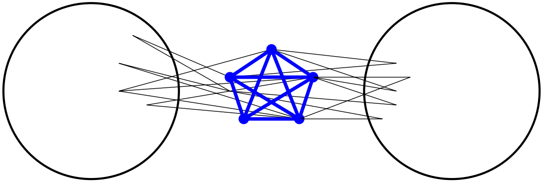


Idea of the proof (*follows classical decomposition technique*)

Let $\boxed{G \in \mathcal{C}_k^{\text{odd}}}$

If G is pyramid-free, then G is a *ring of length k* and possibly *with a universal clique*, or G *has a clique cutset*.

If G contains a pyramid, then:

- consider the *largest blowup of template $\mathbb{T}$* in G;
- study the attachment of every vertex $v \in G \setminus \mathbb{T}$ (if any);
- $N_{\mathbb{T}}[v] = \mathbb{T}$ (i.e., v is a *universal clique*);
  or $N_{\mathbb{T}}[v]$ induces *a clique cutset*.

Proof 2: the class $\mathcal{C}_k^{\text{even}}$, $k \geq 8$

Thm: Structure of $\mathcal{C}_k^{\text{even}}$

Every graph in $\mathcal{C}_k^{\text{even}}$ either is:

- a ring of length k; or
- a blowup of an even template [a];

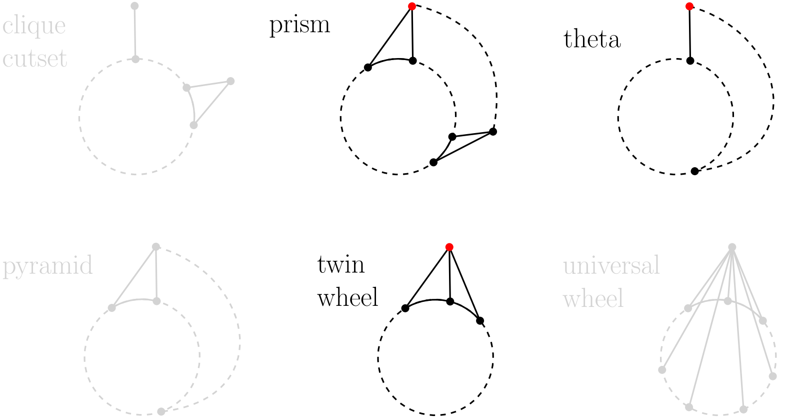
or has either:

- a universal clique; or
- a clique cutset.

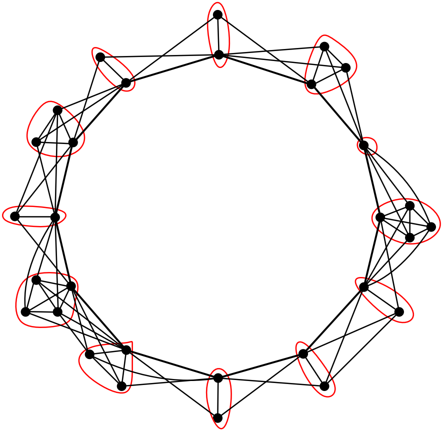
[a] the blowup is similar to those of odd templates

Recall the attachment on holes (for *even k*)

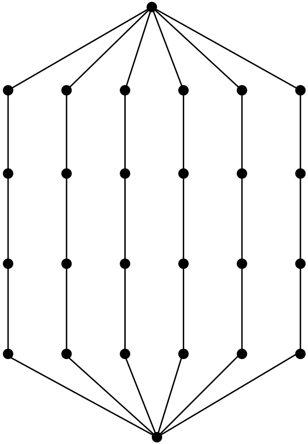
Remark. Suppose that $G \in \mathcal{C}_k^{\text{odd}}$ does not have a *clique cutset* nor a *universal clique*.



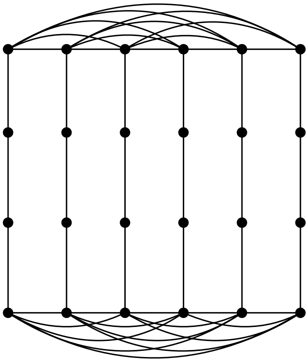
Even rings (when no theta, no prism)



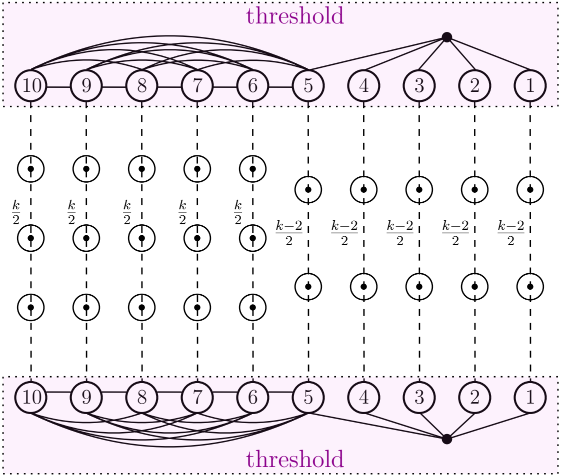
Generalized theta (when no prism)



Generalized prism (when no theta)



Generalization (when thetas or prisms are present): *even* templates



Part 3: Another structural description and algorithmic application

Same structural description but in different formulation

Linda Cook and Paul Seymour independently study this class of graphs.

They describe a structure similar to *template*, and name it **framework**.

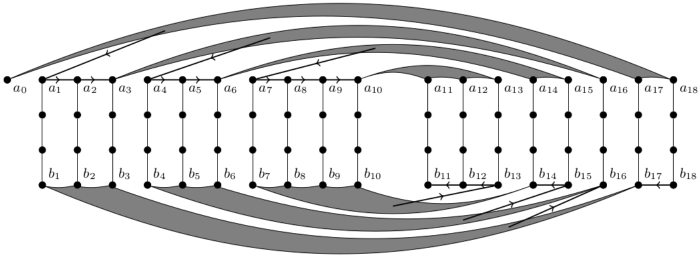


Figure: Structure similar to “template”

Theorem 1 (Horsfield (2022) [4])

The two structural descriptions are equivalent.

Some complexity results on $\mathcal{C}_k^{\text{odd}}$ $(k \geq 7)$

The following results are proved by *Horsfield (2022) [4]*.

‡

1. Given a graph G and odd $k \geq 7$, deciding whether $G \in \mathcal{C}_k$ can be done in $\mathcal{O}(n^8)$.
2. Given weighted $G \in \mathcal{C}_k$, there is an algorithm finding the Maximum Weight Clique in G that works in $\mathcal{O}(n^2 m)$.
3. Given weighted $G \in \mathcal{C}_k$, there is an algorithm finding the Maximum Weight Independent Set in G that works in $\mathcal{O}(n^3 m)$.

Approach: modifying the decomposition theorem by *decomposing the graph using one more cutset*, that is called *modified 2-join*.

‡ $n = |V(G)|$ and $m = |E(G)|$

Tools: 2-join in $\mathcal{C}_k^{\text{odd}}$ (*definition*)

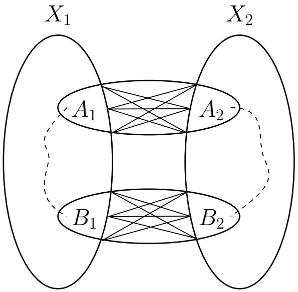


Figure: A partition known as *2-join*, introduced by Cornuéjols and Cunningham (1985)

- (X_1, X_2) is called *2-join partition* of $V(G)$.
- A_1, B_1, A_2, B_2 are nonempty and pairwise disjoint.
- A_i is complete to B_i for $i = 1, 2$.
- There exists a (non-edge) path from A_i to B_i with interior in $X_i \setminus (A_i \cup B_i)$.
- There are no other edges between X_1 and X_2

Tools: 2-join in $\mathcal{C}_k^{\text{odd}}$ (*example*)

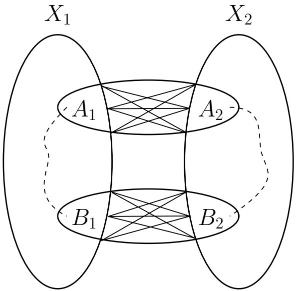
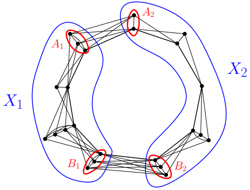
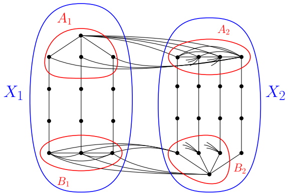


Figure: A partition known as *2-join*, introduced by Cornuéjols and Cunningham (1985)

2-join in a ring:



2-join in a generalized pyramid:



Tools: Modified 2-join in $\mathcal{C}_k^{\text{odd}}$ (Horsfield (2022))

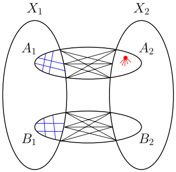


Figure: Modified 2-join of type 1

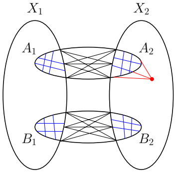


Figure: Modified 2-join of type 2

Theorem 2 (Horsfield (2022))

Every graph in $\mathcal{C}_k^{odd}$ either is:

- *a ring of length k; or*
- *a pyramid;*

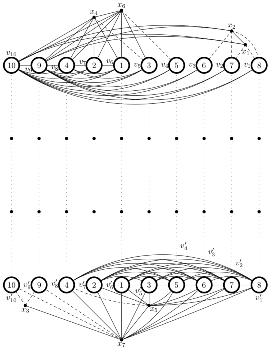
or has:

- *a universal clique; or*
- *a clique cutset; or*
- *a modified 2-join (of type 1 or 2).*

Open questions

1. Algorithmic application using the structural properties of $\mathcal{C}_k^{\text{even}}$
2. Structure of $\mathcal{C}_k$, when $k = 4, 5, 6$
3. Is it possible to generalize the structure theorem into graphs:
    - containing *all odd holes* (i.e., *even-hole-free*)?
    - containing *all even holes* (i.e., *odd-hole-free*)?

Idea: "relaxing" the length of the paths connecting the top part and the bottom part in the blowup of odd-template.



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Thank you for listening ☺☺